

# Generalized Casimir Operators

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## Abstract

Let  $\mathfrak{g}$  be symmetrizable Kac-Moody Lie algebras. In this paper we describe a new class of central operators generalising the Casimir operator. We also prove some properties of these operators and show that these operators move highest weight vectors to new highest weight vectors.

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## Introduction

Let  $\mathfrak{g}$  be symmetrizable Kac-Moody Lie algebra and  $A$  be finitely generated commutative associative algebra with unit. Then  $\mathfrak{g} \otimes A$  is naturally Lie algebra and  $\mathfrak{g}$  is a subalgebra. We consider the category  $\mathcal{O}$  of  $\mathfrak{g} \otimes A$  modules and roughly whose weights are bounded above (see 1.3 for exact definition) Note that any  $\mathfrak{g}$ -module is a  $\mathfrak{g} \otimes A$  module by evaluating at 1. So that the well know category (again denoted by  $\mathcal{O}$ , in  $[K]$ ) of  $\mathfrak{g}$  modules is contained in  $\mathcal{O}$ .

We now construct a class of operators  $\Omega(a, b), a, b \in A$  which act on modules in  $\mathcal{O}$  and commutes with  $\mathfrak{g}$  action. Such operators are called central operators. These operators are variations of Casimir operator and in fact  $\Omega(1, 1)$  is a Casimir operator. It is well known that Casimir operator acts as a scalar on  $\mathfrak{g}$  highest weight vectors. Where as our central operators move one  $\mathfrak{g}$  highest weight vector to another most often. This way if we know one highest weight vector by applying our central operators we can produce more highest weight vectors of the same weight.

The idea of these central operators are born in trying to understand evaluation modules. We will explain this in the simplest case  $A = \mathbb{C}[t, t^{-1}]$ . Let  $V(\lambda_i)$ ,  $1 \leq i \leq n$ , be an irreducible integrable highest weight module for  $\mathfrak{g}$ . Then the tensor product module  $V = \otimes V(\lambda_i)$  is known to be completely reducible as  $\mathfrak{g}$ -module.  $V$  can be made into  $\mathfrak{g} \otimes A$  module by evaluating at distinct points (see 3.3) and is called evaluation module for  $\mathfrak{g} \otimes A$ . In this case there are special central operators denoted by  $\Omega(l, k)$ ,  $1 \leq l, k \leq n$ , which act only on the  $l$ th and  $k$ th components. In fact it is a Casimir operator acting on  $V(\lambda_l) \otimes V(\lambda_k)$  and identity on the rest of the components. Linear combinations of  $\Omega(l, k)$ ,  $1 \leq l, k \leq n$  exhaust all our central operators in the evaluation module case. We have defined highest weight modules  $V(\psi)$  for  $\mathfrak{g} \otimes A$  and all evaluation modules are highest weight modules. But there are many more highest weight modules which are not evaluations modules. We do not know how these central operators act on highest weight modules. When  $\mathfrak{g}$  is simple finite dimensional Lie algebra, the decomposition  $V$  as a  $\mathfrak{g}$  module is a classical open problems. There are several results available for  $n = 2$ . See [KUM] and references therein. But it looks like not much is known for  $n \geq 3$  and here our central operators are very effective. We work out some examples (Examples (3.11) and (3.12)) and note that in these examples that the space spanned by repeatedly applying our central operators on single highest weight vector gives the whole highest weight space of that weight. This will not be true in general. For example  $n = 2$  all our central operators are scalars on  $\mathfrak{g}$ -highest weight vectors and so not very interesting.

In the last chapter we consider  $gl_N$ . It is known that the center of  $U(\mathfrak{gl}_N)$  is finitely generated. In fact for every positive integer  $k$  there is a  $T_k$  (called Gelfand Invariants) in the center of  $U(\mathfrak{gl}_N)$ , but  $T_1, \dots, T_N$  generates the center as an algebra. Now for each  $T_k$  we define a class of central operators (depending on  $A$ ). Denote  $T$  the non-commutative associative algebra generated by all these central operators,  $T$  can be seen as a subalgebra of  $U(\mathfrak{gl}_N \otimes A)$ . As in the earlier case  $T$  can be seen to be independent of  $A$  on an evaluation module as operators. We have proved the following theorem.

Let  $V_1, \dots, V_n$  be irreducible finite dimensional  $gl_N$  modules. Let  $V = \otimes V_i$ . Then  $V$  is completely reducible as  $gl_N \oplus T$ -module (Theorem 4.8)

**Section 1** Throughout the paper all vector spaces and tensor products are over complex numbers  $\mathbb{C}$ .  $U$  always denotes the universal enveloping algebra of a Lie-algebra

(1.1) Let  $\mathfrak{g}$  be a symmetrizable Kac-Moody Lie algebra. Let  $(,)$  be a non-degenerate invariant symmetric bilinear form on  $\mathfrak{g}$ . Let  $\mathfrak{h}$  be a Cartan subalgebra. Let  $\{\alpha_1, \dots, \alpha_l\}$  and  $\{\alpha_1^\vee, \dots, \alpha_l^\vee\}$  be roots and coroots of  $\mathfrak{g}$ . Let  $\Delta$  and  $\Delta^+$  be roots and positive roots of  $\mathfrak{g}$ .

Let

$$\mathfrak{g} = \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \oplus \mathfrak{h}$$

be the root space decomposition of  $\mathfrak{g}$ . See Kac book  $[K]$  for more details.

(1.2) Let  $A$  be finitely generated commutative associative algebra with unit. Denote  $\mathfrak{g}(A) = \mathfrak{g} \otimes A$  with obvious Lie bracket. For any vector space  $V$  denote  $V(A) = V \otimes A$ . Let  $\mathfrak{g} = N^+ \oplus \mathfrak{h} \oplus N^-$  be the standard triangular decomposition. Then  $\mathfrak{g}(A) = N^+(A) \oplus \mathfrak{h}(A) \oplus N^-(A)$  is a triangular decomposition for  $\mathfrak{g}(A)$ . For  $\alpha \in \Delta^+$  define  $ht \alpha = \sum n_i$  where  $\alpha = \sum n_i \alpha_i$ . Note that  $\mathfrak{g} \simeq \mathfrak{g} \otimes 1$ .

(1.3) Definition: A module  $V$  of  $\mathfrak{g}(A)$  is said to be in the category  $\mathcal{O}$  if the following holds

(a)  $V$  is a weight module for  $\mathfrak{g}(A)$  with respect to the Cartan subalgebra  $\mathfrak{h}$  and has finite dimensional weight spaces. (b) For every  $v$  in  $V$  and  $a \in A$  we have  $(X_\alpha \otimes a)v = 0$  for  $ht \alpha \gg 0$  and  $\alpha \in \Delta^+$  and  $X_\alpha \in \mathfrak{g}_\alpha$ .

(1.4) We will now produce a class of irreducible  $\mathfrak{g}(A)$  modules which are in  $\mathcal{O}$ . Let  $\psi : \mathfrak{h}(A) \rightarrow \mathbb{C}$  be any linear map. Consider one dimensional vector space  $\mathbb{C}v$  which is  $N^+(A) \oplus \mathfrak{h}(A)$  module where  $\mathfrak{h}(A)$  acts via  $\psi$  and  $N^+(A)$

acts trivially. Now consider the Verma module.

$$M(\psi) = U(\mathfrak{g}(A)) \bigotimes_{N^+(A) + \mathfrak{h}(A)} \mathbb{C}v.$$

By standard arguments we see that  $M(\psi)$  has a unique irreducible quotient denoted by  $V(\psi)$ . Note that when  $A$  is infinite dimensional  $M(\psi)$  does not have finite dimensional weight spaces.  $V(\psi)$  may have finite dimensional weight spaces depending on  $\psi$ .

Let  $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$  and let  $\mathfrak{h}' = \mathfrak{g} \cap \mathfrak{h}$ . Let  $\mathfrak{h}''$  be any vector space such that  $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''$ . See [K] for more details. Let  $\tilde{\mathfrak{g}} = \mathfrak{g}'(A) \oplus \mathfrak{h}''$ . Lie algebra  $\tilde{\mathfrak{g}}$  was originally considered in [E3] and module theory is developed for the special case  $A$  is a Laurent polynomial algebra in several commuting variables. They have been generalised for any  $A$  in [EB].

**(1.5) Lemma:**  $V(\psi)$  is irreducible as  $\tilde{\mathfrak{g}}$  module.

**Proof** First note that  $U(\tilde{\mathfrak{g}})v = V(\psi)$  as the additional space  $\mathfrak{h}'' \otimes A$  acts as scalars on  $v$ . Suppose  $W$  is a  $\tilde{\mathfrak{g}}$  submodule of  $V(\psi)$ . Let  $w \in W$  be a weight vector of maximal height. Then clearly  $w$  is a highest weight vector in the sense that  $\mathfrak{g}_\alpha w = 0$  for all  $\alpha \in \Delta^+$ . But  $V(\psi)$  does not have highest weight vectors except the multiples of  $v$ . Thus  $w = v$  upto scalar. This proves  $W = V(\psi)$ . Lemma is proved.

Since  $V(\psi)$  is an irreducible  $\tilde{\mathfrak{g}}$  - module, we can use results from [EB].

**(1.6) Proposition (Prop. C, [EB])**  $V(\psi)$  has finite dimensional weight spaces if and only if there exists a co-finite ideal  $I$  of  $A$  such that  $\mathfrak{g}' \otimes A \cdot V(\psi) = 0$ .

**(1.7)** Such  $V(\psi) \in \mathcal{O}$ .

**(1.8)** There exists a special class of co-finite ideals. Fix a positive integer  $n$ . Let  $\mathfrak{m}_i$   $1 \leq i \leq n$ , be distinct maximal ideals of  $A$ . Because of the assumptions on  $A$  we know that  $A/\mathfrak{m}_i \cong \mathbb{C}$ . Consider the co-finite ideal  $I = \bigcap \mathfrak{m}_i$ . Then by Chinese remainder Theorem we have  $A/I \cong \bigoplus \mathbb{C}$  so that  $\mathfrak{g} \otimes$

$A/I \cong \oplus(\mathfrak{g} \otimes A/\mathfrak{m}_i) \cong \oplus\mathfrak{g}$ . For each  $i$  let  $V(\lambda_i)$  be an irreducible highest weight module for  $\mathfrak{g}$  with highest weight vector  $v_i$  and highest weight  $\lambda_i$ . Then  $V = \bigotimes_{i=1}^n V(\lambda_i)$  is a irreducible  $\mathfrak{g}(A)$  module via the surjective map  $\Pi : \mathfrak{g}(A) \rightarrow \oplus\mathfrak{g}$ . Note that the space  $\oplus\mathfrak{h}$  acts as scalars on  $v = v_1 \otimes \dots \otimes v_n$  and now consider the surjective map  $\mathfrak{h}(A) \rightarrow \oplus\mathfrak{h}$ . Let the corresponding map from  $\mathfrak{h}(A) \rightarrow \mathbb{C}$  by  $\psi$ . Then it is easy to see that  $V(\psi) \cong V$  as  $\mathfrak{g}(A)$ -modules.

**(1.9)** Such modules  $V(\psi)$  are called evaluations modules.

Several generalisation of evaluation modules are considered in the literature. See [N] and references there in. For super case see [S].

## 2. Section: Central operators

**(2.1)** We will first recall a certain classical problem in Lie theory. We assume  $\mathfrak{g}$  is simple finite dimensional Lie algebra. Let  $V_1, V_2, \dots, V_n$  be irreducible finite dimensional  $\mathfrak{g}$ -modules. Then  $V = \bigotimes V_i$  be the tensor product module for  $\mathfrak{g}$ . It is well known that  $V$  is completely reducible as  $\mathfrak{g}$ -module.

**(2.2) Open problem : Which  $\mathfrak{g}$  modules occur in  $V$  and with what multiplicity?** There are several results available in the literature and most often for  $n = 2$ . See [KUM] and references there in. We will now define a class of operators, which generalise Casimir operator, acts on the tensor product module  $V$  and commutes with  $\mathfrak{g}$ . The main property of our operators, when applied on a  $\mathfrak{g}$  highest weight vector, produces a new highest weight vector. Whereas the Casimir operator acts as scalar. We will define our operator in the generality of symmetrizable Kac-Moody Lie algebra and they are central operators in the following sense.

**(2.3) Central operators:** A linear operator acting on objects of  $\mathcal{O}$  is called central operator if it commutes with  $\mathfrak{g}$  action.

We will now closely follow Chapter 2 of Kac-book [K]. Let  $\mathfrak{h}^*$  be the

dual of the Cartan subalgebra  $\mathfrak{h}$  and denote the non-degenerate symmetric bilinear form as  $(\cdot, \cdot)$ . We have an isomorphisms

$\nu : \mathfrak{h} \rightarrow \mathfrak{h}^*$  defined by

$$\langle \nu(\mathfrak{h}), \mathfrak{h}_1 \rangle = \nu(\mathfrak{h})(\mathfrak{h}_1) = (\mathfrak{h}, \mathfrak{h}_1)$$

Let  $\rho \in \mathfrak{h}^*$  be such that  $(\rho, \alpha_i) = \frac{1}{2}(\alpha_i, \alpha_i)$ ,  $1 \leq i \leq l$ . Let  $\{e_\alpha^j\}$  be a basis of  $\mathfrak{g}_\alpha$  and let  $\{e_{-\alpha}^j\}$  be the dual basis. Let  $u_1, u_2, \dots, u_l$  be a basis of  $\mathfrak{h}$  and let  $u^1, u^2, \dots, u^l$  be the dual basis. Let  $x(a) = x \otimes a$ ,  $x \in \mathfrak{g}$ ,  $a \in A$ . For  $a, b \in A$ , Define

$$\Omega_{a,b} = \sum_{\alpha \in \Delta^+} \sum_j e_{-\alpha}^j(a) e_\alpha^j(b)$$

Now define the operator

$$(2.4) \quad \Omega(a, b) = 2\nu^{-1}(\rho)(ab) + \sum_i u^i(a) u_i(b) + \Omega_{a,b} + \Omega_{b,a}$$

Certainly  $\Omega(a, b)$  is infinite sum and sits inside some completion of  $U(L(\mathfrak{g}))$ . But  $\Omega(a, b)$  is locally finite on any  $V$  in  $\mathcal{O}$ . (Note that it preserve the weight spaces). In the sense, given a  $v$  in  $V \in \mathcal{O}$  then  $\Omega(a, b)v$  is a finite sum. We also note that  $\Omega(a, b)$  is linear in both variable. That is  $\Omega(\lambda_1 a_1 + \lambda_2 a_2, b) = \lambda_1 \Omega(a_1, b) + \lambda_2 \Omega(a_2, b)$  and the same thing is true in  $b$  also, for  $a_1, a_2, b \in A$ ,  $\lambda_1, \lambda_2 \in \mathbb{C}$ .

**(2.5) conjecture:** We believe that  $\Omega(a, b)$  is a central operator,

**(2.6)** We are not able to prove this. We will note at a later place (3.8) that  $\Omega(a, b)$  is central operator on evaluation modules. See (1.9). We will now prove the conjecture for the special choice of  $A$ .

**(2.7)** Take  $A = \mathbb{C}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$  a Laurent polynomial algebra in  $m$  commuting variables.

**(2.8) Theorem** For  $a, b \in A$ ,  $\Omega(a, b)$  is a central operator on modules in  $\mathcal{O}$ .

We need some notation and lemmas before proving the Theorem. Let  $r = (r_1, r_2, \dots, r_m) \in \mathbb{Z}^m$  and let  $t^r = r_1^{r_1} t_2^{r_2} \dots t_m^{r_m} \in A$ . Recall that  $(, )$  is a non-degenerate bilinear form on  $\mathfrak{g}$ .

It is easy to see that the following bilinear form on  $L(\mathfrak{g})$  is non-degenerate

$$(X \otimes t^r, Y \otimes t^s) = (X, Y) \delta_{r+s, 0}$$

For  $X, Y \in \mathfrak{g}$ ,  $r, s \in \mathbb{Z}^m$ .

Extend the form to the tensor product  $L(\mathfrak{g}) \otimes L(\mathfrak{g})$  by  $(X_1(a_1) \otimes Y_1(b_1), X_2(a_2) \otimes Y_2(b_2)) = (X_1(a_1), X_2(a_2))(Y_1(b_1), Y_2(b_2))$  for  $X_1, Y_1, X_2, Y_2 \in \mathfrak{g}, a_1, a_2, b_1, b_2 \in A$

It is standard fact that the form on the tensor product is non-degenerate.

**(2.9) Lemma:** Let  $a, b \in A$  and let  $\alpha, \beta \in \Delta$ . Let  $z \in \mathfrak{g}_{\beta-\alpha}$ . Then we have

**(2.9.1)**

$$\sum_s e_{-\alpha}^s(a) \otimes [z, e_{\alpha}^s(b)] = \sum_s [e_{-\beta}^s(a), z] \otimes e_{\beta}^s(b)$$

**Proof** By linearity we can assume  $a = t^r$ ,  $b = t^s$ ,  $r, s \in \mathbb{Z}^m$ . It is sufficient to check the pairing both sides with  $e(t^{-r}) \otimes f(t^{-s})$  where  $e \in \mathfrak{g}_{\alpha}$  and  $f \in \mathfrak{g}_{-\beta}$ . Now it is direct checking. See Lemma (2.4) of [K].

**(2.10) Corollary**

$$\sum_s e_{-\alpha}^s(a) [z, e_{\alpha}^s(b)] = \sum_s [e_{-\beta}^s(a), z] e_{\beta}^s(b)$$

**Proof.** Just apply the linear map from  $L(\mathfrak{g}) \otimes L(\mathfrak{g}) \rightarrow U(L(\mathfrak{g}))$  sending  $X \otimes Y \rightarrow XY$ .

**(2.11) Lemma** Let  $a, b \in A$

- (1)  $[\Omega_{a,b}, e_{\alpha_i}] = -\nu^{-1}(\alpha_i)(a) e_{\alpha_i}(b)$
- (2)  $[\Omega_{b,a}, e_{\alpha_i}] = -\nu^{-1}(\alpha_i)(b) e_{\alpha_i}(a)$

**Proof** Proof is similar to the proof of theorem 2.6(a) of [K]. See the second part on page 22. We need to use Corollary (2.10).

**(2.12) Lemma**

$$[\sum u^j(a)u_j(b), e_{\alpha_i}] = \nu^{-1}(\alpha_i)(a)e_{\alpha_i}(b) + e_{\alpha_i}(a)\nu^{-1}(\alpha_i)(b)$$

Direct checking using 2.5.3 of [K]. Also use the fact that  $\nu$  preserves the bilinear form on  $\mathfrak{h}$  and  $\mathfrak{h}^*$ .

**(2.13) Lemma** For  $\alpha \in \Delta$

$$(a) \quad \alpha(\nu^{-1}(\rho)) = (\rho, \alpha)$$

$$(b) \quad \alpha(\nu^{-1}(\alpha)) = (\alpha, \alpha)$$

Just use the 2.5.3 of [K]

## Proof of Theorem (2.8)

From the above Lemma we see that

$$\begin{aligned} [\Omega(a, b), e_{\alpha_i}] &= [2\nu^{-1}(\rho)(ab), e_{\alpha_i}] \\ &+ \nu^{-1}(\alpha_i)(a)e_{\alpha_i}(b) + e_{\alpha_i}(a)\nu^{-1}(\alpha_i)(b) \\ &- \nu^{-1}(\alpha_i)(a)e_{\alpha_i}(b) - \nu^{-1}(\alpha_i)(b)e_{\alpha_i}(a) \end{aligned}$$

Note the first term is equal to

$$2\alpha_i(\nu^{-1}(\rho))e_{\alpha_i}(ab) = 2(\rho, \alpha_i)e_{\alpha_i}(ab)$$

Also note that

$$\begin{aligned} e_{\alpha_i}(a)\nu^{-1}(\alpha_i)(b) - \nu^{-1}(\alpha_i)(b)e_{\alpha_i}(a) &= -\alpha_i(\nu^{-1}(\alpha_i))e_{\alpha_i}(ab) \\ &= -(\alpha_i, \alpha_i)e_{\alpha_i}(ab) = -2(\rho, \alpha_i)e_{\alpha_i}(ab) \end{aligned}$$

Now it is easy to

$$[\Omega(a, b), e_{\alpha_i}] = 0$$

In a similar way we see that  $[\Omega(a, b), e_{-\alpha_i}] = 0$ . Since  $\Omega(a, b)$  zero weight operator it commutes with  $\mathfrak{h}$ . As  $\Omega(a, b)$  commutes with all generators of  $\mathfrak{g}$ , it commutes with  $\mathfrak{g}$ . This completes the proof of the Theorem.



**(2.14) Remark.**

The only obstacle to prove the Conjecture (2.5) is the Lemma (2.9). The rest of the proof goes through.

**3. Section**

**(3.1)** Throughout this section we assume  $A = \mathbb{C}[t, t^{-1}]$  a Laurent polynomial algebra. For any vector space  $V$  we denote  $L(V) = V \otimes A$ .

In this section we give three examples to indicate the importance of our operators. We work with evaluation modules and they have been mentioned in (1.9). In our case they can be made more explicit. First we will simplify our central operators on evaluation modules.

We first recall evaluation modules in the context of  $\mathbb{C}[t_1 t^{-1}]$ . See [E1], [E2] and [E3] for some classification results.

**(3.2)** Let  $\mathfrak{g}$  be symmetrizable Kac-Moody Lie algebra and  $\mathfrak{h}$  be a Cartan subalgebra. Fix a positive integer  $n$  and let  $a_1, a_2, \dots, a_n$  be on-zero distinct complex numbers.

Let  $V(\lambda_1), V(\lambda_2) \dots V(\lambda_n)$  be irreducible highest modules for  $\mathfrak{g}$ . Let  $v_1, v_2, \dots, v_n$  be the corresponding highest weight vectors.

Let  $\underline{\lambda} = (\lambda_1, \lambda_2 \dots \lambda_n)$ ,  $\underline{a} = (a_1, \dots, a_n)$

Let  $V(\underline{\lambda}, \underline{a}) = \otimes_{i=1}^n V(\lambda_i)$ .

Define a  $L(\mathfrak{g})$  module structure on  $V(\underline{\lambda}, \underline{a})$

**(3.3)**  $X \otimes t^k(w_1 \otimes \dots \otimes w_n) = \sum a_i^k w_1 \otimes \dots X w_i \otimes \dots \otimes w_n$  for  $X \in \mathfrak{g}$ ,  $k \in \mathbb{Z}$  and  $w_i \in V(\lambda_i)$ . It can easily checked to be  $L(\mathfrak{g})$ -module. We will now indicate another way of seeing this. Consider the Lie-algebra map

**(3.4)**

$$\Pi(\underline{a}) : L(\mathfrak{g}) \rightarrow \oplus \mathfrak{g}(n \text{ copies})$$

$$\Pi(\underline{a})(X \otimes t^k) = (a_1^k X, \dots, a_n^k X)$$

It is standard fact that  $\Pi(\underline{a})$  is surjective. See [E3]

**(3.5) Claim:**  $V(\underline{\lambda}, \underline{a})$  is an irreducible as  $L(\mathfrak{g})$ -module. First note that  $V$  is an irreducible module for  $\oplus \mathfrak{g}(n \text{ copies})$ . Now using the surjective map  $\Pi(\underline{a})$ ,  $V(\underline{\lambda}, \underline{a})$  becomes  $L(\mathfrak{g})$ -module and one can check that there is precisely one given at (3.3). This proves the claim.

Consider  $\psi(h \otimes t^k) = \sum a_i^k \lambda_i(h)$  which is linear map from  $L(\mathfrak{h})$  to  $\mathbb{C}$ . Recall we have defined an irreducible module  $V(\psi)$  in (1.4). It is easy to see that  $V(\psi) \cong V(\underline{\lambda}, \underline{a})$  as  $L(\mathfrak{g})$ -modules. We will give another proof that  $\Omega(a, b)$  are central operators.

**(3.6)** Let

$$\begin{aligned} P(t) &= \prod_{i=1}^n (t - a_i) \\ P_i(t) &= \frac{\prod_{i \neq j} (t - a_j)}{\prod_{i \neq j} (a_i - a_j)} \end{aligned}$$

It is easy to see

$$\textbf{(3.6.1)} \quad P_i(a_j) = \delta_{ij}$$

$$\textbf{(3.6.2)} \quad \sum P_i(t) = 1$$

We note that  $\mathfrak{g} \otimes P(t)(V(\underline{\lambda}, \underline{a})) = 0$  as it is an evaluation module and  $P(a_i) = 0$  for all  $i$ . Let  $I$  be an ideal generated by  $P(t)$  and  $I$  is a co-finite in  $A$ . Further  $\mathfrak{g} \otimes I V(\underline{\lambda}, \underline{a}) = 0$ . Further we note that  $\ker \Pi = \mathfrak{g} \otimes I$  (See 3.4). Now it is clear that  $\Omega(a, b)$  is zero on  $V(\underline{\lambda}, \underline{a})$  if either  $a \in I$  or  $b \in I$ . We also have  $P_i(t) \notin I$  and is easy to check that the image of  $P_i(t)$ ,  $1 \leq i \leq n$  in  $A/I$  form a basis for  $A/I$ .

Thus to consider  $\Omega(a, b)$ , we can assume  $a$  and  $b$  are linear combinations of  $P_i(t)$ .

**(3.7)** These polynomials  $P_i(t)$  are very special. For example

$$X \otimes P_i(t)(w_1 \otimes \dots \otimes w_n) = w_1 \otimes \dots \otimes X w_i \otimes \dots \otimes w_n$$

Where  $X \in \mathfrak{g}$ ,  $w_i \in V(\lambda_i)$ . So  $X \otimes P(t)$  acts only on the component  $V(\lambda_i)$ . This means  $\Omega(P_i(t), P_i(t))$  acts only on  $i$ th component and it can be seen to be the classical Casimir operator acting on  $V(\lambda_i)$ . In particular it is a

central operator. Similarly  $X \otimes (P_i(t) + P_j(t))(w_1 \otimes \dots \otimes w_n) = w_1 \otimes \dots \otimes Xw_i \otimes \dots \otimes w_n + w_1 \otimes \dots \otimes Xw_j \otimes \dots \otimes w_n$ .

So it will act on  $i$ th and  $j$ th component. Then the operator  $\Omega(P_i(t) + P_j(t), P_i(t) + P_j(t))$  acts only on  $i$ th and  $j$ th component.

It can be readily seen to be classical Casimir operator acting on  $V(\lambda_i) \otimes V(\lambda_j)$ . It is a central operator. Now we have different proof that  $\Omega(P_i(t), P_j(t))$  is a central operator. Now from the last paragraph of (3.6) we know that  $\Omega(a, b)$  is linear combination of  $\Omega(P_i(t), P_j(t))$ . Thus it is another proof that  $\Omega(a, b)$  are central operators on an evaluation module.

We will use these ideas to prove  $\Omega(a, b)$  is central operator for general algebra  $A$  on an evaluation module.

**(3.8)** We will now digress a little to explain evaluation modules in the context of finitely generated commutative associated algebra  $A$  with unit 1. See (1.8) and (1.9) where we have considered evaluation modules of  $\mathfrak{g} \otimes A$ , Let  $\mathfrak{m}_1, \dots, \mathfrak{m}_n$  be distinct maximal ideals and we have  $A/\mathfrak{m}_i \cong \mathbb{C}$ . We also have surjective map  $\Pi : A \rightarrow \oplus A/\mathfrak{m}_i = \oplus \mathbb{C} (n \text{ copies})$ . Consider  $z_i = (0, \dots, 1, \dots, 0) \in \oplus \mathbb{C}$ . Let  $P_i \in A$  such that  $\Pi(P_i) = z_i$ . Then clearly  $P_1, \dots, P_n$  is a basis of  $A \bmod I$  where  $I = \cap \mathfrak{m}_i$ . Consider the corresponding evaluation module  $V(\psi)$  as define in (1.8). It is clear that  $\mathfrak{g} \otimes I \cdot V(\psi) = 0$ . As explained in (3.7) the operators  $\Omega(a, b)$  are linear combination of  $\Omega(P_i, P_j)$ . Again  $\Omega(P_i, P_i)$  and  $\Omega(P_i + P_j, P_i + P_j)$  are standard Casimir operator acting on  $V(\lambda_i)$  and  $V(\lambda_i) \otimes V(\lambda_j)$ . Certainly each of them are central and hence  $\Omega(a, b)$  is a central operator. So as promised in (2.6) we have proved  $\Omega(a, b)$  are central operators on an evaluation module.

**(3.9)** We note that, for an evaluation module, we do not get any new central operators for general  $A$ . It is sufficient to take  $\mathbb{C}[t, t^{-1}]$ .

**3.10 Remark:** Even though the operators on evaluation module case, looks familiar we do not have any evidence that they have been considered by other authors. These operators applied to highest weight vector produce new

highest weight vectors most open. We will explain this with some examples.

**(3.11) Example:** Let  $\mathfrak{g}$  be any symmetrizable Kac-Moody Lie algebra with the standard form  $(\cdot, \cdot)$ . Fix a positive integer  $n$  and consider  $V(\lambda_1), \dots, V(\lambda_n)$  irreducible highest weight modules for  $\mathfrak{g}$  with highest weight vectors  $v_1, \dots, v_n$  and highest weights  $\lambda_1, \lambda_2, \dots, \lambda_n$  which are assumed to be dominant integral. We know that  $V = \otimes_{i=1}^n V(\lambda_i)$  is completely reducible  $\mathfrak{g}$ -module. Put  $\lambda = \sum \lambda_i$ . Let  $V = \oplus_{\beta \geq 0} V_{\lambda - \beta}$  be weight space decomposition. Denote  $V_\mu^+$  be the  $\mathfrak{g}$ -highest weight vectors of weight  $\mu$ .

Let  $\alpha_1^\vee, \dots, \alpha_l^\vee$  be the co-roots. Fix  $j$  and assume  $\lambda_i(\alpha_j) = m_i \geq 1$ . This means  $e_{-\alpha_j} v_i \neq 0 \forall i$ . Let  $w_k = v_1 \otimes \dots \otimes e_{-\alpha_j} v_k \otimes \dots \otimes v_n$ . Let  $z_{k,l} = m_l w_k - m_k w_l \in V_{\lambda - \alpha_j}$ . It is direct checking that  $z_{k,l} \in V_{\lambda - \alpha_j}^+$ . We can see that  $\dim V_{\lambda - \alpha_j} = n$  and  $\dim V_{\lambda - \alpha_j}^+ = n - 1$ . It is easy to see that  $z_{1,2}, \dots, z_{1,3}, \dots, z_{l,n}$  are linearly independent and  $n - 1$  in number. Thus it is a basis for  $V_{\lambda - \alpha_j}^+$ . Since we are working with dual basis in the definition of central operators we see that

$$[e_{\alpha_j}, e_{-\alpha_j}] = \frac{(\alpha_j, \alpha_j)}{2} \alpha_j^\vee$$

Recall the operator  $\Omega(a, b)$  and for simplicity let  $\Omega(l, k) = \Omega(P_l(t), P_k(t))$  for fixed  $l \neq k$ .

The following are direct calculation.

$$(3.11.1) \quad (a) \ i \notin \{l, k\}, \quad \Omega(l, k) w_i = (\lambda_l, \lambda_k) w_i.$$

$$(b) \ \Omega(l, k) w_k = (\alpha_j, \alpha_j) m_k w_l + (\lambda_k - \alpha_j, \lambda_l) w_k.$$

We will now calculate the action of the operators on highest weight vector.

The following is again direct calculation using 3.11.1.

$$(3.11.2) \quad (a) \text{ Let } p \neq q, \ p, q \notin \{l, k\}, \quad \Omega(l, k) z_{p,q} = (\lambda_l, \lambda_k) z_{p,q}.$$

$$(b) \ \Omega(l, k) z_{l,k} = \left( \left( \frac{(\alpha_j, \alpha_j)}{2} (m_l + m_k) \right) - (\lambda_l, \lambda_k) \right) z_{l,k}.$$

$$(c) \ q \neq l, \ k \neq q, \quad \Omega(l, k) z_{k,q} = (\lambda_l, \lambda_k) z_{k,q} - m_q \frac{(\alpha_j, \alpha_j)}{2} z_{k,l}.$$

**3.11.3 Remark** Fix  $k \neq l$ . Then applying central operators repeatedly on  $z_{k,l}$  we get the whole space  $V_{\lambda - \alpha_j}^+$

**(3.12) Example:** Let  $\mathfrak{g}$  be symmetric Kac-Moody Lie-algebra. Let  $\lambda_1, \lambda_2, \dots, \lambda_n$  be dominant integral weights. Let  $V(\lambda_1), \dots, V(\lambda_n)$  be irreducible integrable highest weight modules with highest weight vectors  $v_1, v_2, \dots, v_n$ . Let  $V = \bigotimes_{i=1}^n V(\lambda_i)$  and let  $\lambda = \sum \lambda_i$ . Let  $v = v_1 \otimes \dots \otimes v_n$ . Let  $\{\alpha_1^\vee, \alpha_2^\vee, \dots, \alpha_l^\vee\}$  be co-roots. Since we are assuming  $\mathfrak{g}$  to be symmetric we have

**3.12.1**  $(\lambda_i, \alpha_j) = \lambda_i(\alpha_j^\vee)$  and  $(\alpha_j, \alpha_j) = 2$ . We fix  $j$ .

**3.12.2** We also assume  $m_i = (\lambda_i, \alpha_j) \geq 2$  for all  $i$ . This means  $e_{-\alpha_j}^2 v_i \neq 0$ .

Let  $V = \bigoplus_{\beta \geq 0} V_{\lambda-\beta}$  be the weight space decomposition.  $V_\mu^+$  be the space of  $\mathfrak{g}$ -highest weight vectors. Let  $k \neq l$ .

**3.12.3** Let

$$\begin{aligned} z_{k,l} &= v_1 \otimes \dots \otimes e_{-\alpha_j} v_k \otimes \dots \otimes e_{-\alpha_j} v_l \otimes \dots \otimes v_n. \\ z_k &= v_1 \otimes \dots \otimes e_{-\alpha_j}^2 v_k \otimes \dots \otimes v_n \end{aligned}$$

So that  $z_{k,l}, z_k \in V_{\lambda-2\alpha_j}$

**3.12.4** Let

$$A_{k,l} = 2(m_k - 1)(m_l - 1)z_{k,l} - (m_k - 1)m_k z_l - (m_l - 1)m_l z_k$$

It is direct checking that  $A_{k,l} \in V_{\lambda-2\alpha_j}^+$ . Note that  $A_{k,l} = A_{l,k}$ .

The following is easy to see

**3.12.5**

$$\begin{aligned} (a) \dim V_{\lambda-2\alpha_j} &= \binom{n}{2} + n \\ (b) \dim V_{\lambda-2\alpha_j}^+ &= \binom{n}{2} \\ (c) \#\{A_{k,l}, k \neq l\} &= \binom{n}{2} \end{aligned}$$

and they form a basis for  $V_{\lambda-2\alpha_j}^+$

The following which gives a formula how our operators act on  $V_{\lambda-2\alpha_j}^+$ . As earlier let  $\Omega(k.l) := \Omega(P_k, P_l)$ . Let  $k \neq l$ .

**3.12.6** (a)  $p \neq q, p, q \notin \{k, l\}, \Omega(p, q)A_{k,l} = (\lambda_p, \lambda_q)A_{k,l}$

(b)  $q = k, p \neq l, \Omega(p, q)A_{ql} = (\lambda_p, \lambda_q - \alpha_j)A_{ql} - \frac{(m_l-1)m_l}{(mp-1)}A_{p,q} + \frac{(m_q-1)m_q}{(mp-1)}A_{p,l}$

(c)  $q = k, p = l, \Omega(p, q)A_{p,q} = (\lambda_p - \alpha_j, \lambda_q - \alpha_j)A_{p,q} - (m_q + m_p)A_{p,q}$

**3.12.7** Let  $\Omega$  be the non-commutative associative algebra generated by  $\Omega(l, k), 1 \leq l, k \leq n$ . Then for a fixed  $k \neq l$ .

$$\{\Omega \cdot A_{k,l}\} = V_{\lambda-2\alpha_j}^+.$$

**(3.13)** We recall some well known facts from the representation theory of  $\mathfrak{sl}(2, \mathbb{C})$  found in Humphreys' book [H]. Let  $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$  with basis  $x, y, h$  and  $[x, y] = h, [h, x] = 2x$  and  $[h, y] = -2y$ . Let  $m$  be a fixed positive integer and let  $V(m)$  denote the finite dimensional irreducible highest weight module for  $\mathfrak{sl}(2, \mathbb{C})$  with highest weight vector  $v$ . Then

**3.13.1**  $hv = mv, y^m v \neq 0, y^{m+1} v = 0$ . In Humphrey's book [H], Lemma 2.6.2 states

**(3.13.2)**  $xy^a = y^a x + ay^{a-1}(h - a + 1)$ . for all  $a \in \mathbb{N}$ .

For  $m, n \in \mathbb{N}$  one has the Clesbch-Gordan decomposition theorem

**(3.13.3)**  $V(m) \otimes V(n) \cong V(m+n) \oplus V(m+n-2) \oplus \cdots \oplus V(|m-n|)$  and this decomposition is multiplicity free.

Let us write down the highest weight vectors (up to a scalar) in this decomposition in terms of tensor products of weight vectors from  $V(m)$  and  $V(n)$ . Let  $v_1$  and  $v_2$  be the highest weight vectors of  $V(m)$  and  $V(n)$  respectively. Then the highest weight vector of weight  $w_1$  of weight  $m+n-2l$  is a linear combination of the vectors  $y^i v_1 \otimes y^{l-i} v_2$  where  $0 \leq i \leq l$ . Then

**(3.13.4)**  $w_l = \sum_{i=0}^l a_i y^i v_1 \otimes y^{l-i} v_2$  with  $a_i \in \mathbb{C}$ . As  $w_l$  is a highest weight vector we have

$$0 = xw_l = \sum_{i=0}^l x a_i y^i v_1 \otimes y^{l-i} v_2 + \sum_{i=0}^l a_i y^l v_1 \otimes xy^{l-i} v_2$$

Thus one concludes

$$(3.13.5) \quad i(m-i+1)a_i + (l-i+1)(n-l+i)a_{i-1} = 0$$

for  $1 \leq i \leq l$ . One can solve this recursion relation to obtain that the vector  $(a_0, a_1, \dots, a_l)$  is uniquely determined by just one of the coefficients say  $a_0$  and each of the  $a_i$  are nonzero.

Let  $m > n > k$  and  $m - n > k > 0$ . Our goal now is to see how  $V(m) \otimes V(n) \otimes V(k)$  decomposes and using the operators  $\Omega(b_i, b_j)$  how to obtain a basis for all of the highest weight vectors in this tensor product. The following matrix will explain this decomposition:

$$\begin{pmatrix} V(m+n+k) & V(m+n+k-2) & \cdots & V(m+n-k) \\ V(m+n+k-2) & V(m+n+k-4) & \cdots & V(m+n-k-2) \\ \vdots & \vdots & \ddots & \vdots \\ V(m+n+k-2l) & V(m+n+k-2l-2) & \cdots & V(m+n-k-2l) \\ \vdots & \vdots & \ddots & \vdots \\ V(m-n+k+2) & V(m-n+k) & \cdots & V(m-n-k+2) \\ V(m-n+k) & V(m-n+k-2) & \cdots & V(m-n-k) \end{pmatrix}$$

Let  $V_{ij} := V(m+n-2i+k-2j)$ . The matrix  $(V_{ij})_{0 \leq i \leq n, 0 \leq j \leq k}$  is the matrix above with  $n+1$  rows and  $k+1$  columns. Notice that sum of the elements in the  $(l+1)$ -st row is nothing but the decomposition of  $V(m+n-2l) \otimes V(k)$ . Let  $i+j=l$  with  $0 \leq i \leq n, 0 \leq j \leq k$ , then  $V_{ij} \cong V(m+n+k-2l)$ . The set of  $V_{ij}$ ,  $i+j=l$  is what we will call the *anti-diagonal* and they are all isomorphic.

Set  $s_l = \min(l, k)$  and  $d_l = s_l + 1$  Then define

(3.13.6)  $d'_l = \#\{(i, j) \mid i+j=l, 0 \leq i \leq n, 0 \leq j \leq k\}$  It is easy to see that the following are true:

$$d'_l = d_l, \quad 0 \leq l \leq n, \quad d'_{n+i} = k+1-i, \quad 0 \leq i \leq k.$$

Just for clarity we see  $d'_n = k+1$  and  $\min(n, k) = k$ .

(3.13.7) Notice that the first entry of the  $(l+1)$ -row is the top component of  $V(m+n-2l) \otimes V(k)$ . The highest weight vector of this component is  $w_l \otimes v_3$  where  $v_3$  is the highest weight vector of  $V(k)$ .

Recall  $w_l = \sum_i a_i y^i v_\otimes y^{l-i} v_2$  and each summand is nonzero. In particular  $w_l \otimes v_3$  has  $v_1 \otimes y^l v_2 \otimes v_3$  as a summand.

**(3.13.8)** Let  $P_i = b_i$ . We claim  $w_l \otimes v_3$  and  $\Omega(b_2, b_3)(w_l \otimes v_3)$  are linearly independent. To prove the claim first note that  $x(b_2)y(b_3)$  occurs in  $\Omega(b_2, b_3)$ . Thus  $\Omega(b_2, b_3)(w_l \otimes v_3)$  contains the term

$$v_1 \otimes y^{l-1} v_2 \otimes y v_3$$

and this term doesn't occur in  $w_l \otimes v_3$ . Now the claim follows.

**(3.13.9)** The following set contains exactly  $d_l$  linearly independent vectors for  $l \leq n$ .

$$\{\Omega(b_2, b_3)^j(w_l \otimes v_3) \mid 0 \leq j \leq s_l\}$$

**Proof:** Note that by argument similar to the above we see that

$$\Omega(b_2, b_3)^j(w_l \otimes v_3)$$

contains the summand  $v_1 \otimes y^{l-j} v_2 \otimes y^j v_3$  which doesn't occur for lower  $j$ . Thus the set consists of linearly independent vectors. This completes the proof of the claim.

**(3.13.10)** Note that for  $j > k$ , the summands  $v_1 \otimes y^{l-j} v_2 \otimes y^j v_3$  is zero as  $y^{k+1} v_3 = 0$ . Similarly for  $j > l$  that the term doesn't make sense. Thus  $j$  can go only up to  $\min(l, k) = s_l$ .

Up to now we have only worked with highest weight vectors  $w_l \otimes v_3$ ,  $0 \leq l \leq n$ . There are exactly  $n+1$  highest weight vectors in the first column of the matrix. By applying operators  $\Omega(b_2, b_3)$  we get all of the highest weight vectors of the corresponding anti-diagonal.

**(3.13.11)** Now we will work with the highest weight vectors of the last row and prove that by applying  $\Omega(b_2, b_3)$  repeatedly we can obtain all other highest weight vectors. Next consider the last row. The first entry in the



last row is taken care of. We will only work with the second entry of the last row which is the representation

$$V_{n,1} = V(m - n + k - 2).$$

Note that this module is the second component of  $V(m - n) \otimes V(k)$ . Since  $w_n$  is the highest weight vector of  $V(m - n)$  it is easy to see that

$$z = (kyw_n \otimes v_3) - (m - n)w_n \otimes yv_3$$

is the highest weight vector of the second component of  $V(m - n) \otimes V(k)$ . Recalling the definition of  $w_n$ , we see that

$$v_1 \otimes y^n v_2 \otimes yv_3$$

is a nonzero summand of  $z$  where we use the fact that  $m > n$ .

By applying  $\Omega(b_2, b_3)^j$ , with  $j \leq k$  to  $z$  we see that

$$v_1 \otimes y^{n-j} v_2 \otimes y^j v_3$$

is a summand of  $\Omega(b_2, b_3)^j z$ . They are linearly independent and there are  $k$  in number. This is precisely the number of modules in the anti-diagonal as  $d'_{n+1} = k$ . This argument breaks down for  $j \geq k + 1$  as  $y^{k+1} v_3 = 0$ . Similarly the argument is valid for the other entries in the last row and we leave the details to the reader.

We will summarise the above results. We have taken the highest weight vectors of the first column and the last row. Then we have applied our operators to the highest weight vectors and obtained all other highest weight vectors.

#### 4. Section

**(4.1)** In this section we consider general linear algebra  $\mathfrak{g} = gl_N$  for a fixed positive integer  $N$ . Let  $A$  be any commutative associated algebra with unit 1. Then  $\mathfrak{g} \otimes A$  is a naturally Lie algebra. We will now define vectors in

$U(\mathfrak{g} \otimes A)$  which commutes with  $\mathfrak{g} \cong \mathfrak{g} \otimes 1$ . They are automatically central operators on  $\mathfrak{g} \otimes A$  modules. Let

$$\{E_{ij}, 1 \leq i, j \leq N\}$$

be the standard basis with Lie bracket.

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{kj}$$

**(4.2)** For a positive integer  $k$  and  $b_1, b_2, \dots, b_k$  in  $A$ , define

$$T_k(b_1, b_2, \dots, b_k) = \sum_{(i_1, i_2, \dots, i_k)} E_{i_1 i_2}(b_1) E_{i_2 i_3}(b_2) \dots E_{i_k i_1}(b_k)$$

where  $(i_1, i_2, \dots, i_k)$  run over all possible indices.

Let  $Z$  be the center of  $U(\mathfrak{g})$ . Then it is well known that  $T_k(1, \dots, 1) \in Z$  for all  $k$ .

**(4.3) Fact:** It is a classical result of Harishchandra that  $T_1(1, \dots, 1), \dots, T_N(1, \dots, 1)$  generate  $Z$  as an algebra.

**(4.4) Proposition:** Notation as above

$$[T_k(b_1, \dots, b_k), \mathfrak{g}] = 0$$

**Proof** Clearly  $T_k(b_1, \dots, b_k) \in U(\mathfrak{g} \otimes A)$ . Note that  $T_k(b_1, b_2, \dots, b_r + b_r^1, \dots, b_k) = T_k(b_1, b_2, \dots, b_r, \dots, b_k) + T_k(b_1, b_2, \dots, b_r^1, \dots, b_k)$

Let  $E_{j_1 j_2} \in \mathfrak{g}$ . For  $1 \leq r \leq k$ .

Define

$$B_r = \sum_{(i_l, \dots, i_k)} E_{i_1 i_2}(b_1) \dots E_{i_{r-1} i_r}(b_{r-1}) E_{i_r j_2}(b_r) E_{j_1 i_{r+2}}(b_{r+1}) \dots E_{i_k i_1}(b_k)$$

$$B_r^1 = \sum_{(i_l, \dots, i_k)} E_{i_1 i_2}(b_1) \dots E_{i_{r-1} j_2}(b_{r-1}) E_{j_1 i_{r+1}}(b_r) E_{i_{r+1} i_{r+2}}(b_{r+1}) \dots E_{i_k i_1}(b_k)$$

Now it is direct checking that

$$z = [T_k(b_1, \dots, b_k), E_{j_1 j_2}] = \sum_{r=1}^k (B_r - B_r^1)$$

Notice that  $B_r = B_{r+1}^1$  for  $1 \leq r \leq k-1$  and  $B_k = B_1^1$ .

Now it is easy to see that  $z = 0$ . This proves the proposition.

There are several open problems. Let  $T$  be the non-commutative associative algebra generated by  $T_k(b_1, \dots, b_k)$  for all possible  $b_1, \dots, b_k$  in  $A$  and  $k \in \mathbb{Z}$ .

Let  $\overline{Z} = \{X \in U(\mathfrak{g} \otimes A) \mid [X, \mathfrak{g}] = 0\}$ .

Then we know that  $T \subset \overline{Z}$

**(4.5) Problem** 1) Does equality holds?

2) Does  $T$  finitely generated?

**(4.6)** In the rest of the section we take  $A = \mathbb{C}[t, t^{-1}]$ . Fix positive integer  $n$ . Let  $\underline{\lambda} = (\lambda_1, \dots, \lambda_n)$  be dominant integral weights. Let  $\underline{a} = (a_1, \dots, a_n)$  be non-zero distinct real numbers. We need this assumption as the form we consider later is conjugate linear. In any case, these  $a_s$  are not important as our interest is  $V(\underline{\lambda}, \underline{a})$  as  $\mathfrak{g}$ -module.

For each  $i$ , let  $V(\lambda_i)$  be an irreducible finite dimensional highest weight module with highest weight vector  $v_i$  for  $\mathfrak{g}$ . Consider  $V(\underline{\lambda}, \underline{a}) = \bigotimes_{i=1}^n V(\lambda_i)$  is an irreducible evaluation module for  $\mathfrak{g} \otimes A$ . Recall from earlier section the polynomials  $P_1(t), \dots, P_n(t)$  such that  $\sum_{i=1}^n P_i(t) = 1$

**(4.7) Remark.**  $T_k(1, \dots, 1)$  which is a central operator and acts as scalar on every  $\mathfrak{g}$ -component of  $V(\underline{\lambda}, \underline{a})$ . But  $T_k(1, 1, \dots, 1)$  splits into several operators.  $T_k(P_{i_1}(t), \dots, P_{i_k}(t))$  where each operator does not act as scalars (most often). For clarity we write one such operator. Take  $n = 4$  and consider  $V(\lambda_1) \otimes \dots \otimes V(\lambda_4)$ . Take  $k = 3$  and

$$T_3(P_1(t), P_2(t), P_3(t))w_1 \otimes \dots \otimes w_4 = \sum_{(i_1, i_2, i_3)} E_{i_1 i_2} w_1 \otimes E_{i_2 i_3} w_2 \otimes E_{i_3 i_1} w_3 \otimes w_4$$

Notice that there is no action on  $w_4$ . We believe such operators are completely new.

**(4.8) theorem)** The finite dimensional module  $V(\underline{\lambda}, \underline{a})$  is completely reducible as  $T \oplus \mathfrak{g}$ -module.

**Proof** Let  $w^\vee$  be an anti-linear anti-automorphism on  $U(\mathfrak{g})$  such that  $w^\vee(E_{ij}) = E_{ji}$ . It is well known that each  $V(\lambda_i)$  admits a unique Hermitian contravariant form which is positive definite such that  $(v_i, v_i) = 1$ . Here contravariant means  $(E_{ij}w_1, w_2) = (w_1, E_{ji}w_2)$  for  $w_1, w_2 \in V(\lambda_i)$ . Now define positive definite Hermitian form on the tensor product  $V(\underline{\lambda}, \underline{a})$  by the following

$$(w_1 \otimes \dots \otimes w_n, u_1 \otimes u_2 \dots u_n) = \prod (w_i, u_i)$$

It is easy to check that

$$(E_{ij} \otimes t^l(w_1 \otimes \dots, w_n), u_1 \otimes u_2 \otimes \dots u_n) = (w_1 \otimes \dots \otimes w_n, E_{ji} \otimes t^l(u_i \otimes u_2 \dots \otimes u_n))$$

We have used the fact that  $a_i$ 's are real which is assumed in (4.6)

$$\begin{aligned} & (T_k(b_1, \dots, b_k)(w_1 \otimes w_2 \otimes \dots w_n), u_1 \otimes u_2 \otimes \dots u_n) \\ &= ((w_1 \otimes w_2 \otimes \dots \otimes w_n), T_k(b_k, b_{k-1} \dots b_1)(u_1 \otimes u_2 \dots \otimes u_n)) \end{aligned}$$

To prove the theorem it is sufficient to prove that  $V_\mu^+$  is completely reducible as  $T$ -module. (Recall that  $V_\mu^+$  is the space of  $\mathfrak{g}$ -highest weight vectors and as  $T$  commutes with  $\mathfrak{g}$ , so  $V_\mu^+$  is a  $T$ -module).

Since  $V_\mu^+$  is finite dimensional it has irreducible  $T$  submodules. Let  $W$  be the sum of such  $T$  submodules. It is sufficient to prove that  $W = V_\mu^+$ . Consider

$$W^\perp = \{u \in V_\mu^\perp \mid (u, w) = 0 \ \forall w \in W\}$$

It is standard fact that  $W \cap W^\perp = \{0\}$  and  $W \oplus W^\perp = V_\mu^+$ . See [G]

**Claim**  $W^\perp$  is  $T$  submodule. Let  $u \in W^\perp$  and consider -

$$\begin{aligned} & (T_k(b_i, \dots, b_k)u, w) \\ &= (u, T_k(b_k, b_{k-1}, \dots, b_1)w) \\ &= 0 \end{aligned}$$

This proves that  $T_k(b_1, \dots, b_n)w \in W^\perp$  and hence the claim is proved.

Now  $W^\perp$  cannot be non-zero as it contradicts the maximality of  $W$ . Thus  $W^\perp = \{0\}$  and  $W = V_\mu^+$ . This completes the proof of the theorem.

**(4.9) Remarks** We know  $V = \oplus W(\lambda)$  where  $W(\lambda)$  is sum of all isomorphic  $\mathfrak{g}$ -modules. Now each  $W(\lambda)$  is a  $T$ -module. The open problem is whether  $W(\lambda)$  is  $\mathfrak{g} \oplus T$  irreducible.

**(4.10)** We will now extend the above results for the orthogonal and symplectic Lie algebras. We will only sketch the results and leave the details to the reader. We will follow closely Alexander Molev's book [Chapter 4 of [Y]]. These Lie algebras can be treated as subalgebras of  $gl_N$ . As in the book we will number the rows and columns of  $N \times N$  matrices by the indices

$$\{-k, \dots, -1, 0, 1, \dots, k\}$$

if  $N = 2k + 1$  (orthogonal case) and by

$$\{-k, \dots, -1, 1, \dots, k\}$$

if  $N = 2k$ . (symplectic case).

Define

$$\theta_{ij} = \begin{cases} 1 & \text{in the orthogonal case} \\ (\text{sign } i \text{ sign } j) & \text{in the symplectic case} \end{cases}$$

Note that  $\theta_{ij}^2 = 1$ ,  $\theta_{ij} = \theta_{ji}$  and  $\theta_{ij} \theta_{jk} = \theta_{ik}$ .

Let  $\mathfrak{g}_N$  denotes the one of these Lie-algebras

Define

$$F_{ij} = E_{ij} - \theta_{ij} E_{-j, -i}$$

**(4.11)** The following is direct verification

$$[F_{ij}, F_{kl}] = \theta = \delta_{jk} F_{il} - \delta_{il} F_{kj} + \delta_{l, -j} \theta_{ij} F_{k, -i} - \delta_{i, -k} \theta_{ij} F_{-j, l}$$

As earlier let  $A$  be commutative associative algebra with unit. We will now construct central operators for the Lie algebra  $\mathfrak{g}_N \otimes A$ . Let  $b_1, b_2, \dots, b_k \in A$  and  $k > 0$ .

**(4.12) Define**

$$S_k(b_1, \dots, b_k) = \sum_{(i_1, \dots, i_k)} F_{i_1 i_2}(b_1) \dots F_{i_k i_1}(b_k)$$

It is direct checking that the above operators are central. Let  $S$  be the non-commutative associative algebra generated by the operators at (4.12).

Suppose  $V_1, V_2, \dots, V_n$  be irreducible finite dimensional modules for  $\mathfrak{g}_N$ .  
Let  $V = \bigotimes_{i=1}^n V_i$ .

**(4.13) Theorem**  $V$  is completely reducible as  $S \oplus \mathfrak{g}_N$ -module.

The proof is similar to the proof of Theorem 4.8.

## References

- [EB] S. Eswara Rao and Punita Batra, Classification of irreducible integrable highest weight modules for current Kac-Moody Lie algebra, preprint.
- [EF] S. Eswara Rao and V. Futorny, Representations of loop Kac-Moody Lie algebras, *Comm. Algebra* 41(2013), No. 10, 3775-3792.
- [EFS] S. Eswara Rao, V. Futorny and Sachin Sharma, Weyl modules associated to Kac-Moody lie algebras. Accepted in *Comm. Algebra*.
- [E1] S. Eswara Rao, On representations of loop algebras, *Comm. Algebras*, 21(1993), No.6, 2131-2153.
- [E2] S. Eswara Rao, Classification of irreducible integrable modules for multi-loop algebras with finite dimensional weight spaces, *Journal of Algebra*, 246 (2001), 215-225.
- [E3] S. Eswara Rao, Classification of irreducible integrable modules for toroidal Lie algebras with finite dimensional weight spaces, *Journal of Algebra*, 277 (2004), 318-348.
- [G] Werner Greub, *Linear algebra*, Springer-Verlag, New York-Berlin fourth edition, 1975, Graduate Texts in Mathematics, No.23.
- [GE] I.M. Gelfand, Center of the infinitesimal group ring, *Math. Sbornik* 26 (1950), 103-112 (Russian) English transl. in I.M. Gelfand, 'Collected papers', Vol. II: Springer-Verlag, 1988, PP 22-30.
- [H] J.E. Humphreys, *Introduction to Lie algebras and representation theory*, Vol. 9 of Graduate Texts in Mathematics, Springer Verlag, New York 1978.
- [K] V.G. Kac, *Infinite dimensional Lie algebras*, Cambridge university press, Cambridge, 3rd edition 1990.
- [KUM] S. Kumar, Tensor product decomposition, proceedings of International Congress of Mathematicians, Hyderabad, India Vol. III, 1226-1261, New Delhi, 2010.

- [M] A. Molev, Yangians and classical Lie algebra, SURU-143, AMS(2007).
- [NS] E. Neher and A. Savage, A survey of equivariant map algebras with open problems, Recent developments in algebraic and combinatorial aspects of representation theory, Contemp. Math. 602, 165-182, AMS, Providence RI, 2013.
- [PRV] K.R. Parthasarathy, R. Ranga Rao and V.S. Varadarajan, Representation of complex semi-simple Lie groups and Lie algebras, Ann. of Math. 85 (1967), 383-429
- [S] A. Savage, Equivariant map superalgebras, Math. Z, 277 (2014) No.1-2, 373-399.

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